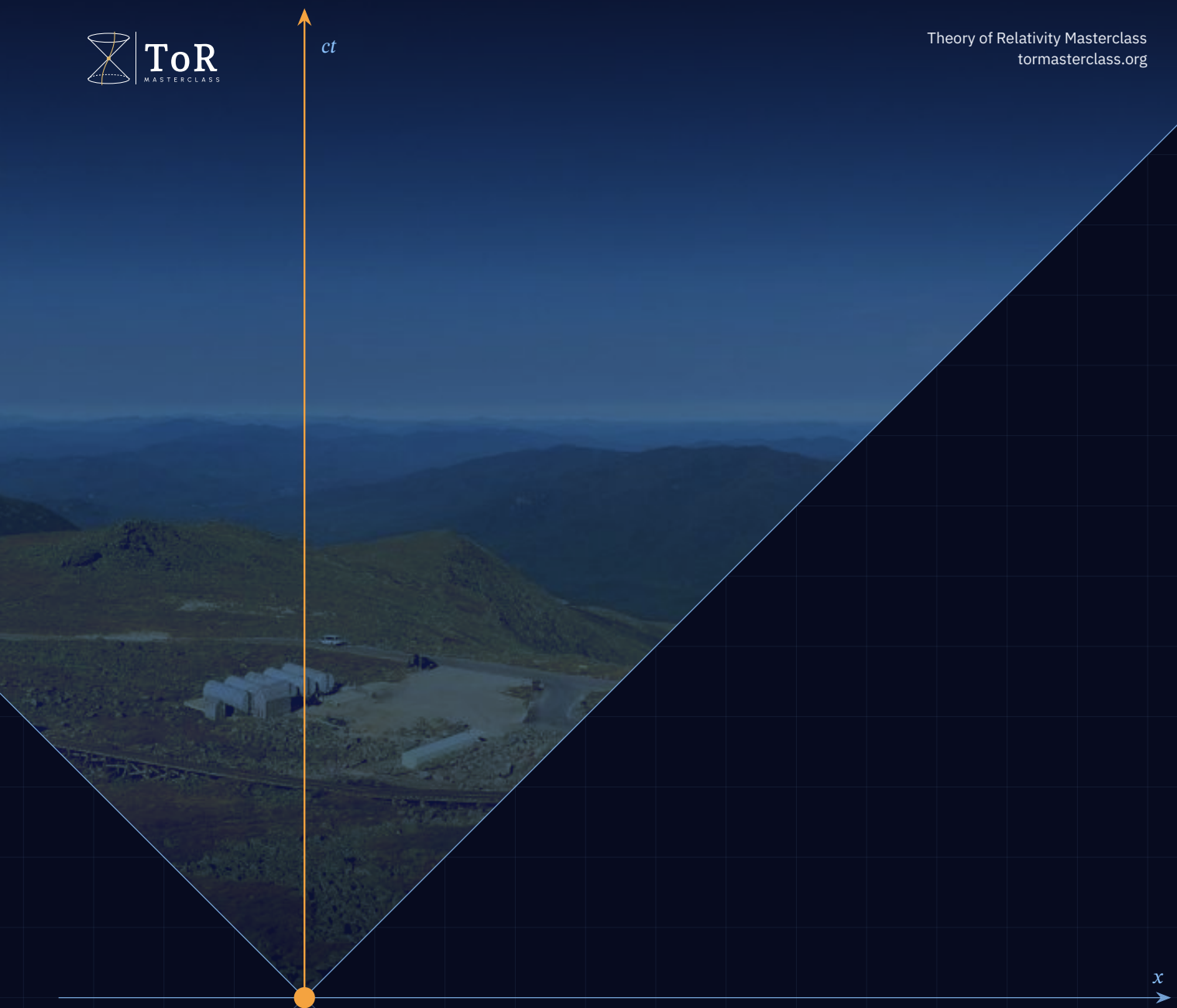


UNIT 3

Simultaneity, Time Dilation, and Length Contraction

Why moving clocks run slow and moving rulers shrink, derived from a light clock

Image: Famartin, Wikimedia Commons, CC BY-SA 4.0

UNIT 3

By the end of this unit, I should be able to answer these questions:

- Why do two observers in relative motion disagree about whether two distant events happened at the same moment?
- How does a light clock show that a moving clock ticks more slowly, and by what factor?
- Why do muons made about 15 km up in the atmosphere reach the ground when their lifetime at rest is only 2.2 microseconds?
- Why is a moving ruler shorter along its direction of motion but unchanged across it?
- What is proper time, and which observer measures it?

In 1962 two physicists drove a detector up Mount Washington in New Hampshire, counted the muons that arrived at the summit each hour, and then counted again in their laboratory in Cambridge, Massachusetts, 1,907 m lower down. Muons are unstable particles that cosmic rays create about 15 km up in the atmosphere. At rest they live 2.2 microseconds on average, and even at the speed of light a muon covers only about 660 metres in that time, so by the decay law of Unit 1 only a few dozen of the 563 muons counted at the top each hour should have survived the trip down. David H. Frisch (1918-1991) and his colleague James H. Smith found 408. The same shortfall had been seen in a rougher form by Bruno Rossi (1905-1993) and David B. Hall in Colorado in 1940. The explanation is the subject of this unit: a moving clock runs slow, and a muon is a clock. The factor by which it runs slow follows from the two postulates of Unit 2 with nothing more than Pythagoras' theorem, and the same factor shortens every moving ruler.

1 The relativity of simultaneity

Unit 2 defined an observer as a whole frame of reference, with a clock at every point and all clocks synchronized by light signals. Two events are **simultaneous** in that frame when the clocks at the two places show the same reading. There is an equivalent test that needs no clocks at all: if light flashes from the two events meet exactly at the midpoint between them, they were emitted at the same time, because light covers the two equal distances at the same speed. The second postulate says that this speed is c in every inertial frame. The consequence is that two frames in relative motion do not agree on which events are simultaneous, and the argument that shows it is Einstein's own.

Consider a long train that moves along a straight embankment at a speed v to the right, as drawn in Figure 1. Two lightning bolts strike the two ends of the train, the rear end A and the front end B, and each leaves a burn mark on the train and a second one on the embankment. An observer M stands on the embankment exactly midway between the two marks there, and a passenger M' sits exactly midway between the two marks on the train. The strikes happen at the moment when M' passes M, so the two are side by side as the flashes set out.

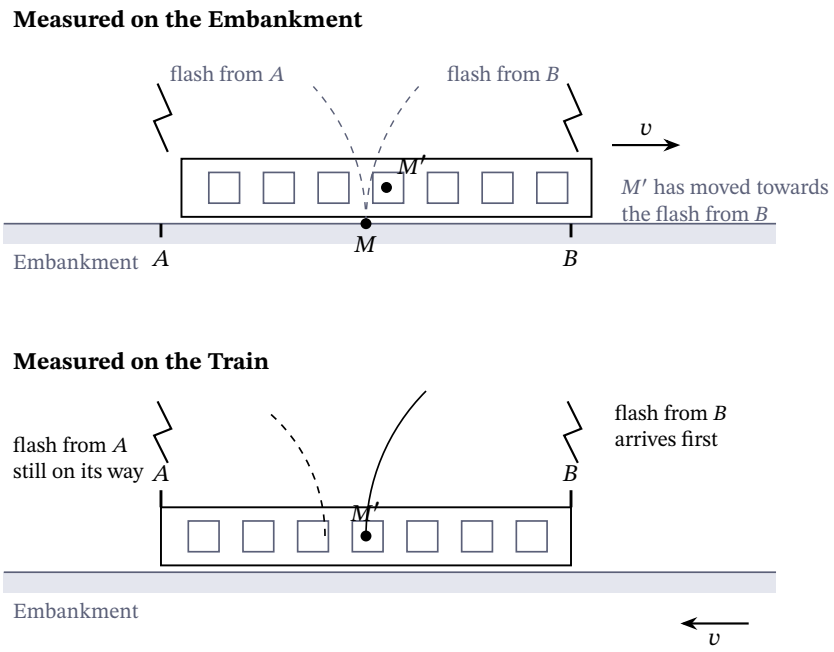


Figure 1. The two lightning strikes seen from the embankment (top) and from the train (bottom). On the embankment the two flashes meet at the midpoint M, so the strikes were simultaneous there. The passenger M' moves towards the flash from B and meets it first; measured in the train, where both flashes also travel at c over equal distances, the strike at B happened before the strike at A.

For the observer on the embankment the two flashes travel at c over equal distances and arrive at M at the same instant, so the strikes were simultaneous in the frame of the embankment. The passenger, meanwhile, has moved, because between the strikes and the arrival of the light M' travelled a short distance to the right, towards the flash from B and away from the flash from A. The flash from B therefore reaches M' before the flash from A does. Up to here the embankment observer and the passenger agree, because the order in which two flashes reach one person is a fact that everybody can check.

Now take the same facts into the frame of the train. The passenger is at rest midway between the two marks on the train, and by the second postulate the two flashes travel at c in the train's frame as well, so each needs the same time to cover the same distance to M'. The flash from B arrived first, so it must have set out first. Measured in the train, the lightning struck B before it struck A. Neither observer has made a mistake and neither has a privileged view. The two frames assign different times to the two events, and both assignments are correct in their own frame.

EXCURSION

Einstein's train and embankment

ALBERT EINSTEIN (1879-1955) wrote one book for readers without a physics degree, Über die spezielle und die allgemeine Relativitätstheorie (gemeinverständlich), which means On the Special and the General Theory of Relativity, Generally Comprehensible. He finished it in December 1916, and the publisher Vieweg in Braunschweig brought it out in 1917. The English translation by Robert W. Lawson followed

in 1920 and is still in print. Einstein asked of the reader only a school education in mathematics and a good deal of patience. The thought experiment with the railway train, the embankment, and the two lightning strikes fills its ninth section, The Relativity of Simultaneity, and the argument above follows his.

KEY EQUATION

Relativity of simultaneity

Two events that are simultaneous in one inertial frame are not simultaneous in a second inertial frame that moves along the line joining them. Events that happen at the same place are simultaneous for everyone or for no one. The disagreement grows with the speed of the second frame and with the distance between the events, and it is far too small to notice at everyday speeds.

Three remarks keep the result in proportion.

1. Two events at the same place and the same time are a single happening as far as timing is concerned, and every frame agrees about them. The disagreement needs a separation between the events along the direction of motion.
2. The order of the two strikes can also be reversed by a third frame moving the other way. If the order of two events depends on the frame, one may fear that an effect could come before its cause in some frame; Unit 5 shows why this can never happen for two events that are able to influence each other.
3. The size of the disagreement is set by v/c and by the length of the train, and Unit 4 derives the formula. For a 300 m train at 100 m/s the two frames disagree by a third of a picosecond, where a picosecond is a millionth of a microsecond, which is why nobody noticed before 1905.

CHECK YOUR UNDERSTANDING

Question 1

In Einstein's thought experiment the two lightning strikes are simultaneous for the observer on the embankment. The train moves towards the front end B. In which order do the strikes happen as measured in the frame of the train?

(A) The strike at the front end B happens first

(B) The strike at the rear end A happens first

(C) They are simultaneous in the train frame as well

(D) The order depends on where in the train the passenger sits

The solution is at the end of the unit.

2 The light clock

The simplest clock to analyse in physics is a pulse of light bouncing between two parallel mirrors. The mirrors are a distance d apart, and one **tick** is the round trip of the pulse from the lower mirror to the upper one and back, which takes the time

$$\Delta\tau = \frac{2d}{c}.$$

With $d = 15$ cm the round trip is 30 cm, and since light covers about 30 centimetres per nanosecond, which is a billionth of a second, the clock ticks once per nanosecond. The Greek letter τ (tau) instead of t marks the time that this clock itself shows, and the reason for the separate letter appears in a moment. Nothing in the argument that follows depends on the clock being made of light; any other clock placed next to it must agree with it, because otherwise the clock that falls out of step would reveal the motion of the frame, which the first postulate forbids.

2.1 The clock in motion

Now set the clock on a train that moves at speed v to the right, with the mirrors arranged so that the pulse bounces at right angles to the motion, and watch it from the platform (Figure 2). Between the departure of the pulse from the lower mirror and its return there, the clock has moved a distance $v\,\Delta t$, where Δt is the time the platform clocks measure for one tick. The pulse has to reach the upper mirror where that mirror is at the time of the reflection, which is halfway along the distance $v\,\Delta t$, so from the platform its path consists of two slanted legs instead of one vertical line of length $2d$, and it is longer than $2d$. In Newtonian physics that would not matter, because the light would move faster to make up for it, as a ball thrown upwards inside a moving train does, since the ball carries the train's forward speed with it. The second postulate removes this possibility: on the platform too, the pulse moves at exactly c. A longer path at the same speed takes a longer time, and so the moving clock ticks more slowly as judged from the platform.

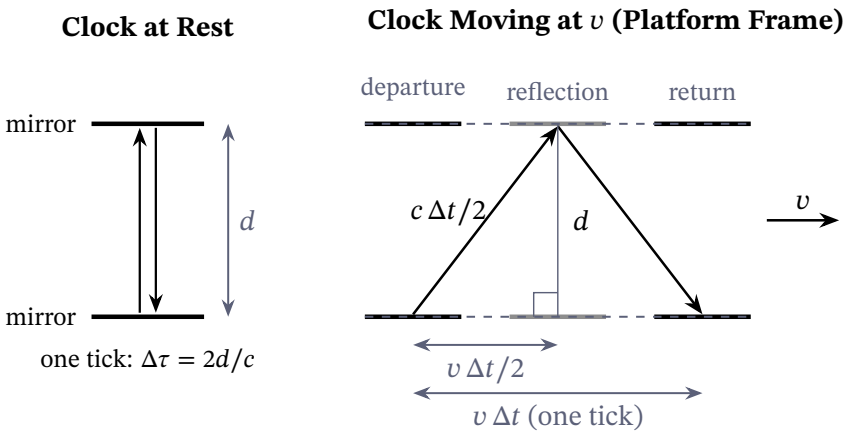


Figure 2. A light clock at rest (left) and the same clock moving at speed v to the right (right), drawn in the platform frame at the departure, the reflection, and the return of the pulse. Each slanted leg is the hypotenuse of a right-angled triangle whose other sides are the mirror distance d and half the distance $v\,\Delta t$ travelled by the clock during one tick.

DERIVATION

Time dilation from the light clock

In the platform frame one tick lasts Δt, and during that time the clock moves $v\,\Delta t$. Each slanted leg of the light path is the hypotenuse of a right-angled triangle with the vertical side d and the horizontal side $v\,\Delta t/2$. The pulse covers each leg in the time $\Delta t/2$ at the speed c, so each leg is $c\,\Delta t/2$ long. Pythagoras' theorem for one triangle gives

$$\left(\frac{c\,\Delta t}{2}\right)^2 = d^2 + \left(\frac{v\,\Delta t}{2}\right)^2 .$$

Multiply by 4, collect the terms with Δt^2 on the left, and solve:

$$\Delta t^2\,(c^2 - v^2) = 4d^2 \qquad \Longrightarrow \qquad \Delta t = \frac{2d}{\sqrt{c^2 - v^2}} = \frac{2d}{c}\,\frac{1}{\sqrt{1 - v^2/c^2}}.$$

The first factor is the tick $\Delta\tau = 2d/c$ of the clock at rest, so $\Delta t = \Delta\tau/\sqrt{1 - v^2/c^2}$. The square root is smaller than 1 whenever $v > 0$, which makes Δt larger than $\Delta\tau$. Nothing in the derivation depends on d or on the clock being made of light.

The factor that appears here appears in every formula of special relativity, and it has a name and a letter of its own, the Greek γ (gamma).

KEY EQUATION

The Lorentz factor

$$\gamma = \frac{1}{\sqrt{1 - v^2/c^2}}$$

The **Lorentz factor** of a speed v is 1 at rest, stays within half a per cent of 1 up to a tenth of the speed of light, and grows without limit as v approaches c. It is never smaller than 1.

KEY EQUATION

Time dilation

$$\Delta t = \gamma\,\Delta\tau$$

A clock that moves at speed v through an inertial frame shows the

time $\Delta\tau$ between two of its ticks, while the synchronized clocks of that frame measure the longer time Δt between the same two ticks. Moving clocks run slow by the Lorentz factor.

The name **time dilation** describes the effect from the platform's point of view, since to dilate means to stretch or widen, and the interval between the ticks of the moving clock is stretched. Table 1 gives the Lorentz factor for the speeds this program uses, and Figure 3 shows its shape. Up to a tenth of the speed of light nothing noticeable happens; between a half and nine tenths of c the factor grows from 1.15 to 2.3; and in the last per cent it climbs without limit. The binomial approximation of Unit 1 describes the flat part. For speeds v much smaller than c, written $v \ll c$,

$$\gamma \approx 1 + \frac{v^2}{2c^2},$$

which is the form used for everything that moves at aircraft or satellite speeds. The two speeds $0.6c$ and $0.8c$ give the Lorentz factors $5/4$ and $5/3$ exactly, because 3, 4, and 5 form a Pythagorean triple, three whole numbers with $3^2 + 4^2 = 5^2$, and the worked examples of this and the next units use them for that reason.

Speed v	Lorentz Factor γ	Example
$0.1c$	1.005	no everyday object comes close
$0.5c$	1.155	
$0.6c$	1.25	exactly $5/4$
$0.8c$	1.667	exactly $5/3$
$0.9c$	2.294	
$0.99c$	7.09	
$0.995c$	10.0	the muons of Section 3
$0.999c$	22.4	

Table 1. The Lorentz factor at the speeds used in this program, computed from $\gamma = 1/\sqrt{1 - v^2/c^2}$ and rounded to the figures shown.

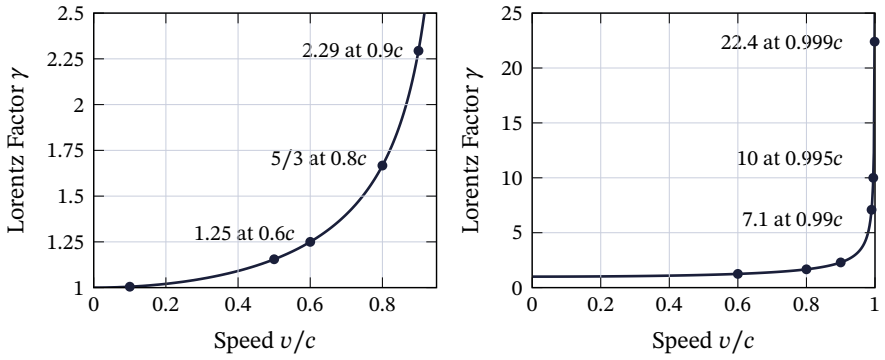


Figure 3. The Lorentz factor against the speed as a fraction of c, computed from its definition. The left panel shows the slow growth up to $0.9c$, where γ has only reached 2.3; the right panel continues to $0.999c$, where it is 22.4. The dots mark the values of Table 1.

WORKED EXAMPLE

A light clock on a train at $0.6c$

A light clock with mirrors $d = 3.00$ m apart rides on a train at $v = 0.6c$. On the train one tick is the round trip of 6.00 m, which takes $\Delta\tau = 6.00\,\text{m}/c = 20.0$ ns. From the platform, each half tick lasts a time $\Delta t/2$ in which the clock moves $0.6c\,\Delta t/2$ and the pulse moves $c\,\Delta t/2$ along the hypotenuse. Pythagoras' theorem gives $(c\,\Delta t/2)^2 = (3.00\,\text{m})^2 + (0.6\,c\,\Delta t/2)^2$, and since $1-0.36 = 0.64$ this is $0.64\,(c\,\Delta t/2)^2 = 9.00\,\text{m}^2$, which gives $c\,\Delta t/2 = 3.75$ m. The pulse travels 3.75 m to the upper mirror and 3.75 m back, 7.50 m in all, while the clock advances $0.6 \times 3.75\,\text{m} = 2.25$ m in each half tick, which is $2 \times 2.25\,\text{m} = 4.50$ m in all. At the speed c the 7.50 m take 25.0 ns, so one tick of the moving clock lasts $\Delta t = 25.0$ ns on the platform clocks, against 20.0 ns on the train, and the ratio $25.0/20.0 = 1.25$ is the Lorentz factor $5/4$ of Table 1.

INTERACTIVE

The light clock in motion



Set the speed of the right-hand clock with the slider, or pick one of the speeds of the table, and watch its light pulse trace the slanted path while the clock at rest ticks beside it. The readouts give the Lorentz factor, the length of the pulse's path per tick on the platform, and how many ticks the moving clock completes while the clock at rest completes ten. Notice that at a tenth of the speed of light the two clocks stay in step for a long while, and that beyond 0.9c the moving clock falls behind at once.

Click the picture or this link to try it in the online version of this unit.

2.2 Proper time

The two symbols in the time dilation formula are not interchangeable, and keeping them apart is the single most useful habit this unit can teach. The **proper time** $\Delta\tau$ between two events is the time measured by the one clock that is present at both events. The two ticks of the light clock both happen at the clock, so the clock's own reading is the proper time between them. The platform frame needs two different clocks to time the same two ticks, one where the first tick happens and one farther along where the second tick happens, and it reads off Δt as the difference between them. Whenever a problem asks which time is which, the question to ask is which clock was at both events. That clock shows the proper time, and every other frame measures a longer time.

The proper time is also a time that has a meaning independent of any frame. Every inertial observer who computes the proper time between two ticks of a clock from their own Δt and their own v gets the same $\Delta\tau$, because that is the number written on the clock's face. Unit 5 builds the whole geometry of spacetime on this quantity.

2.3 Who sees whose clock run slow

The first postulate makes the situation symmetric. From the train, the platform is what moves, at v to the left, and a light clock on the platform ticks slowly by the same factor γ as judged from the train. Each observer finds the other's clock slow. Students who meet this for the first time usually suspect a contradiction, and the reason there is none deserves a close look.

When the platform compares its clocks with the train's clock, it uses two synchronized platform clocks at two places and one train clock that passes both. When the train compares, it uses two synchronized train clocks and one platform clock that passes both. The two comparisons involve different pairs of events, so they are not the same measurement with the roles swapped, and there is no one pair of clocks, one on each side, about which the two sides make opposite claims. What makes the two results compatible is Section 1: the two platform clocks are synchronized in the platform frame, but in the train frame they are not, and the train reads the platform's claim as the result of comparing one clock against two that were set wrongly. The platform says the same about the train's clocks. Both are right, and the difference between them is exactly the relativity of simultaneity, which is why that section came first. Unit 6 turns this symmetry into the twin paradox, where the two clocks finally meet again and one of them really is behind.

CHECK YOUR UNDERSTANDING

Question 2

A clock A stands on a station platform and a clock B rides on a train passing at $0.6c$. Which statement is right?

(A) Only B runs slow, because the train is the one that is moving

(B) The station finds B running slow, and the train finds A running slow

(C) Only A runs slow, because the platform moves relative to the train

(D) Each frame finds the other's clock running slow, so the effects cancel and both tick at the same rate

The solution is at the end of the unit.

WORKED EXAMPLE

A day on the International Space Station

The International Space Station orbits at $v = 7.66\,\mathrm{km\,s^{-1}}$, which is $v/c = 2.56 \times 10^{-5}$. The Lorentz factor differs from 1 only in the tenth decimal place, and the binomial approximation gives that small difference directly:

$$\gamma - 1 \approx \frac{v^2}{2c^2} = \frac{(7.66 \times 10^3)^2}{2\,(3.00 \times 10^8)^2} = 3.26 \times 10^{-10}.$$

In one day of 86,400 s on the ground, the clocks on the station fall behind by $86{,}400\,\mathrm{s} \times 3.26 \times 10^{-10} = 2.82 \times 10^{-5}$ s. The station's day is shorter by about 28 microseconds, which adds up to about a hundredth of a second in a year. This is the velocity part of the effect alone. The weaker gravity at the station's altitude makes its clocks run faster by a smaller amount, and Unit 8 computes it; for the satellites of the GPS, much higher up, the gravitational part is the larger of the two.

3 Time dilation measured

A formula derived from a thought experiment deserves a real one. Three measurements are described here, in order of precision: muons falling through the atmosphere, hydrogen ions glowing in a tube, and lithium ions circulating in a storage ring.

3.1 Muons from cosmic rays

Cosmic rays, mostly protons, hit the nuclei of the air about 15 km up and produce showers of short-lived particles. Among the products are muons, heavy relatives of the electron with a mass of $105.7\,\mathrm{MeV}/c^2$, about 207 times the mass of the electron, which decay with a mean lifetime of 2.2 microseconds when at

rest. Many of them travel close to the speed of light. Without time dilation, a muon at $0.995c$ would move on average $0.995 \times 299{,}792\,\text{km}\,\text{s}^{-1} \times 2.2\,\mu\text{s} \approx 660$ m before decaying, and the 15 km of atmosphere would be more than twenty mean lifetimes deep, since $15{,}000\,\text{m}/660\,\text{m} \approx 23$. The fraction surviving a time t is $e^{-t/\tau}$ for a mean lifetime τ, as for any radioactive decay (the lifetime carries the letter τ because it is a proper time, the time on the muon's own clock), so about e^{-23}, or one in ten billion, would reach the ground. In fact muons are the most common charged particles from cosmic rays at sea level, and about one passes through the palm of a hand every second.

Frisch and Smith turned this into a measurement, which they published in the American Journal of Physics in 1963. Their detector was a block of plastic scintillator, a material that gives a flash of light when a charged particle stops in it, with a stack of metal absorber plates above it. The plates slowed the muons, and only those in a narrow band of speeds, between $0.9950c$ and $0.9954c$ on arrival at the summit, came to rest inside the scintillator rather than above or below it, because faster muons passed through the block and slower ones stopped in the plates. A stopped muon decays after a while into an electron or a positron and two neutrinos, and the electron makes a second flash, so each muon that stopped was recorded together with its decay time. At the summit of Mount Washington, 1,917 m above sea level, they registered 563 ± 10 such muons per hour, and the distribution of decay times gave the mean lifetime at rest of 2.2 microseconds directly, since a stopped muon is at rest in the detector. Then they repeated the measurement in Cambridge, 1,907 m lower, with a slightly thinner stack of plates so that the same muons, slowed by the air to about $0.988c$, would again stop in the scintillator. Figure 4 shows the two counts next to the predictions with and without time dilation.

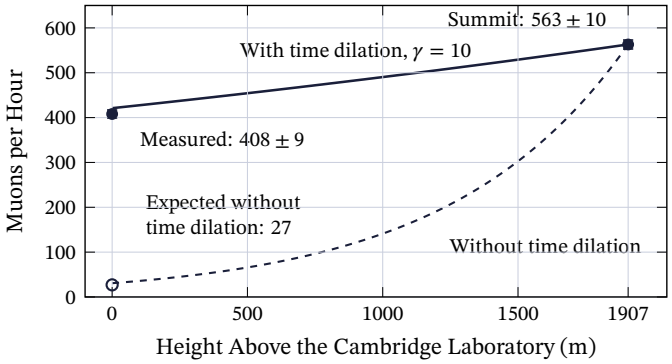


Figure 4. The muon counts of Frisch and Smith at the summit of Mount Washington and in Cambridge, 1,907 m lower. The solid curve is the survival expected with time dilation at $0.995c$, the dashed curve the survival expected without it, both computed for a mean lifetime at rest of 2.2 microseconds and a constant speed. The open circle is the paper's own prediction without time dilation, which allows for the muons slowing in the air. Data: Frisch and Smith (1963), American Journal of Physics 31, 342.

WORKED EXAMPLE

The muons of Frisch and Smith

In the frame of the ground the trip from the summit to the laboratory covers 1,907 m at about $0.995c$, which takes

$$\Delta t = \frac{1{,}907\,\text{m}}{0.995 \times 2.998 \times 10^{8}\,\text{m}\,\text{s}^{-1}} = 6.39\,\mu\text{s}.$$

Without time dilation the surviving fraction would be $e^{-6.39/2.2} = e^{-2.9} = 0.055$, and 563 muons per hour at the top would become 31 per hour at the bottom. Frisch and Smith expected 27, because the muons slow down a little in the air, and a slower muon covers less ground in each lifetime, so more of them decay on the way. With time dilation the muon's own clock, which is the one present at both the summit and the laboratory, advances only by the proper time

$$\Delta\tau = \frac{\Delta t}{\gamma} = \frac{6.39\,\mu\text{s}}{10.0} = 0.64\,\mu\text{s},$$

since $\gamma = 10.0$ at $0.995c$. The surviving fraction is $e^{-0.64/2.2} = 0.75$, and 563 per hour become 421 muons per hour at the laboratory. The measured rate was 408 ± 9 per hour, fifteen times the Newtonian prediction. Expressed as a time dilation factor, that is, as the Lorentz factor that turns 563 into 408 by the decay law, the two counts give 8.8 ± 0.8, and the authors' own prediction, which allows for the muons slowing from $0.995c$ to $0.988c$ on the way down, was 8.4 ± 2.

EXCURSION

Rossi and Hall on Mount Evans

The muon was discovered in cosmic rays in 1936 and was known for a decade as the mesotron, because its mass lies between those of the electron and the proton. In 1940 Bruno Rossi (1905-1993), who had left Italy after the racial laws of 1938 and had arrived in the United States the year before, took counters to Echo Lake in Colorado, 3,240 m up on the road to the summit of Mount Evans, and to Denver at 1,616 m. With David B. Hall he compared, for mesotrons of different momenta, which for particles of one mass means different speeds, how many disappeared on the way down, and found that the faster ones decayed less per metre of descent, in the way that time dilation demands and by about the right amount. From the loss they derived a mean lifetime at rest of about 2.4 microseconds, close to the modern 2.2. The paper, Variation of the Rate of Decay of Mesotrons with Momentum, appeared in the Physical Review in 1941, and it was the first measurement of time dilation that did not depend on a spectral line, the method of Ives and Stilwell described in the next subsection. The particle itself changed names twice more: after the pion was found in 1947 the mesotron became the mu meson, and once it was clear that it was a heavy electron rather than a meson, a name that had meanwhile been given to a different family of particles, the one the pion belongs to, it became the muon. The mountain changed its name as well, to Mount Blue Sky, in 2023. Rossi moved to the Massachusetts Institute of Technology in 1946, the institution from which Frisch set out for Mount Washington sixteen years later.



Figure 5. Bruno Rossi, whose measurements of 1940 in Colorado were the first to show time dilation with cosmic-ray particles. Credit: Massachusetts Institute of Technology, CC BY-SA 3.0.

3.2 Moving atoms as clocks

An atom emits light at frequencies fixed by its internal structure, so a beam of moving atoms is a set of moving clocks, and the frequency of their light read in the laboratory is a reading of those clocks. The first experiment of this kind was done by HERBERT E. IVES (1882-1953) and his colleague G. R. Stilwell at the Bell Telephone Laboratories, and published in the Journal of the Optical Society of America in 1938 under the title An Experimental Study of the Rate of a Moving Atomic Clock. They accelerated hydrogen molecular ions, hydrogen molecules with one electron missing, through voltages of up to 18,350 V, let them pick up electrons and glow, and photographed the blue-green spectral line Hβ of hydrogen at 486.1 nm, one of the few sharp colours in which hydrogen gas glows, looking along the beam and, by means of a small mirror, against it at the same time. The ordinary Doppler shift, the change of frequency of a moving source that is familiar from the pitch of a passing siren and which Unit 5 treats in full, moves the line to shorter wavelengths in one view and to longer ones in the other by the same amount, and the average of the two views cancels it. What remains in the average is a shift of the whole pattern to longer wavelengths, because the moving atom's clock runs slow and emits a lower frequency, and this shift is the measurement.

WORKED EXAMPLE

The shift of Ives and Stilwell

A hydrogen molecular ion H_2^+ has a mass of $2 \times 1.66 \times 10^{-27}$ kg, and 18,350 V give it the kinetic energy $eV = 1.602 \times 10^{-19}\,\text{C} \times 18{,}350\,\text{V} = 2.94 \times 10^{-15}$ J. From $\frac{1}{2}mv^2 = eV$, which is accurate enough at this speed, $v = 1.33 \times 10^6\,\text{m s}^{-1}$, which is $v/c = 4.44 \times 10^{-3}$. The Lorentz factor exceeds 1 by $v^2/(2c^2) = 9.85 \times 10^{-6}$. The emitted frequency is lower by this fraction, so the wavelength in the average of the two views is longer by

$$\Delta\lambda = \lambda_0\,(\gamma - 1) = 486.1\,\text{nm} \times 9.85 \times 10^{-6} = 4.8 \times 10^{-3}\,\text{nm}.$$

The predicted shift is 0.0048 nm, or 0.048 Å in the ångström, a tenth of a nanometre, used by the 1938 paper. At their highest voltage Ives

and Stilwell measured 0.047 Å. A difference of five thousandths of a nanometre, on a plate on which the two Doppler-shifted lines sat several nanometres apart, was at the limit of what a spectrograph could do, and Ives himself, who never accepted Einstein's theory, read his result as a confirmation of Lorentz's older ether theory. The number is the same in both.

The lithium ions of the research spotlight carry the same idea to a third of c and to a precision of a few parts in a billion.

RESEARCH SPOTLIGHT

Gerald Gwinner *University of Manitoba*

Stored lithium ions confirm the Lorentz factor to a few parts in a billion

In the ESR storage ring of the GSI Helmholtz Centre for Heavy Ion Research in Darmstadt, lithium ions ($^7Li^+$) circulate at $v/c = 0.338$, for which the Lorentz factor is 1.0625, and keep circulating for long enough to be probed with lasers. Gerald Gwinner of the University of Manitoba and sixteen colleagues, most of them at institutes in Germany, used a transition inside the ion, a jump of one of its electrons between two energy levels, as the clock, as Ives and Stilwell had used the hydrogen line, and aimed one laser along the beam and a second one against it. Each laser is in resonance with the moving ions at a frequency shifted by the ordinary Doppler effect, in opposite directions for the two lasers, and by the slowing of the ions' clock, in the same direction for both. The product of the two resonance frequencies equals the square of the ion's rest frequency exactly if the clock rate follows the Lorentz factor of Section 2, whatever the speed. By driving both transitions at once in a scheme that removes the spread of ion speeds, the team measured both frequencies to better than four parts in a billion and found the relation satisfied to within 2.3 parts in a billion. It is the most precise test of time dilation of the Ives-Stilwell kind, and it also bounds theories in which the laws of physics would single out a preferred frame.

Read the paper: Botermann et al. (2014), Test of Time Dilation Using Stored Li+ Ions as Clocks at Relativistic Speed, Physical Review Letters 113, 120405

4 Length contraction

4.1 The muon's point of view

Return to the muon, but this time ride with it. In its own frame the muon is at rest and lives 2.2 microseconds on average, with no time dilation to help it, because its own clock is the one that shows the proper time. What it sees coming is Mount Washington, rushing up at $0.995c$. From the ground frame we know that the muon's clock advanced by 0.64 microseconds between the summit and the laboratory, and during 0.64 microseconds a mountain approaching at $0.995c$ covers only $0.64\,\mu\text{s} \times 0.995 \times 2.998 \times 10^8\ \text{m s}^{-1} = 191$ m. Both frames agree that the muon was at the summit and later at the laboratory, so in the muon's frame the vertical distance between the two is 191 m, a tenth of the 1,907 m measured on the ground. In its own frame the muon lives its ordinary 2.2 microseconds, and what has shrunk is the mountain. The two descriptions are the same physics seen from two frames, and the factor between them is the Lorentz factor once more.

DERIVATION

Length contraction from time dilation

A rod of length L_0 lies at rest in the frame S along the x axis, and a clock moves along it at speed v. In S the clock passes the rod's two ends at two events separated by the time $\Delta t = L_0/v$, measured by two synchronized clocks of S at the two ends. The moving clock is present at both events, so it shows the proper time $\Delta\tau = \Delta t/\gamma$ between them. In the clock's own frame S' the clock is at rest and the rod slides past it at speed v, taking the time $\Delta\tau$ from one end to the other. The length of the rod in S' is therefore

$$L = v\,\Delta\tau = \frac{v\,\Delta t}{\gamma} = \frac{L_0}{\gamma}.$$

The clock measured the rod's length with one clock and no ruler, by timing it, and found it shorter by the Lorentz factor.

KEY EQUATION

Length contraction

$$L = \frac{L_0}{\gamma} = L_0\sqrt{1 - \frac{v^2}{c^2}}$$

A rod of **proper length** L_0, its length measured in the frame in which it is at rest, has the shorter length L in a frame through which it moves at speed v along its own direction. Lengths at right angles to the motion do not change.

The length of a moving rod means something definite: it is the distance between the positions of its two ends at the same time in the measuring frame. Since frames disagree about what "at the same time" means, they can disagree about the length without contradiction, and Section 1 is once more the reason. The proper length L_0 is the length everybody can agree on, in the same way that the proper time is the time everybody can agree on, because it is the one measured in the rod's rest frame, and any other frame measures less. Like time dilation, the effect is symmetric. A metre stick on the platform is $1/\gamma$ metres long as measured from the train, and a metre stick on the train is $1/\gamma$ metres long as measured from the platform.

WORKED EXAMPLE

A train and a platform at $0.8c$

A train of proper length 100 m, measured by its crew with a tape, passes a platform of proper length 250 m at $0.8c$, for which $\gamma = 5/3$. The station staff mark the positions of the two ends of the train at one instant of station time and find the train to be $100\text{ m}/\gamma = 60.0$ m long. The crew do the same for the platform at one instant of train time and find it $250\text{ m}/\gamma = 150$ m long. So the station measures a train of 60.0 m, while the crew measure a platform of 150 m, and both keep their own proper lengths. The staff can also time the train: its 60.0 m pass a lamp post in $60.0\text{ m}/(0.8c) = 0.250\ \mu$s on the one station clock at the post. The crew find that the post takes $100\text{ m}/(0.8c) = 0.417\ \mu$s to pass their whole train, because for them the post moves at $0.8c$ along a train that has its full proper length. The two times agree with Section 2: the front and the rear pass the post at the same place in the station frame, so the station clock reads the proper time $0.250\ \mu$s, and the crew, who need a clock at each end of the train, measure γ times as much.

CHECK YOUR UNDERSTANDING

Question 3

A spaceship has a length of 120 m according to its crew and passes a space station at $0.8c$. Which lengths does each side measure for the ship?

(A) The crew measure 72 m and the station measures 120 m

(B) Both measure 120 m, because the ship's length is a property of the ship

(C) The crew measure 120 m and the station measures 72 m

(D) The crew measure 120 m and the station measures 200 m

The solution is at the end of the unit.

A length contraction that takes a 100 m train to 60 m needs a speed of 240,000 kilometres per second, and there is no everyday counterpart. An airliner at 250 m/s is shorter in flight by $L_0 - L \approx L_0 v^2/(2c^2)$, the binomial approximation of Unit 1 applied to the square root, which for a 70 m fuselage is 2.4×10^{-11} m, a quarter of the diameter of a hydrogen atom. Where the speed is large enough for length contraction to matter, a rule of thumb for the whole subject applies. The muon's 191 m is the same statement as its 0.64 microseconds, and a problem that looks hard in one frame is often the time dilation problem of the other frame in disguise.

5 Lengths across the motion and what a camera sees

5.1 The two rulers

The key equation of Section 4 claims that only lengths along the motion change. The claim can be proved by an argument that needs no formula, only the first postulate and a paint brush. Take two identical metre sticks, A and B, held upright, at right angles to the x axis, and let B move past A along x at speed v so that their lower ends pass at the same height (Figure 6). Fix a brush to the top of B so that it marks A as it passes.

Suppose that motion shortened lengths across the direction of motion. Measured in the frame of A, the moving stick B would then be shorter than a metre, and its brush would leave a mark on A below A's top. But measured in the frame of B, it is A that moves, so A would be the shorter one, and B's brush would pass above A's top and leave no mark at all. A mark on a stick is a fact, either there or not, and two frames cannot disagree about it. The assumption must therefore be wrong, and the same argument rules out a lengthening. Lengths across the motion are the same in every frame. The argument cannot be run along the direction of motion, because there the two ends pass each other at different times, and the relativity of simultaneity gives the frames room to disagree.

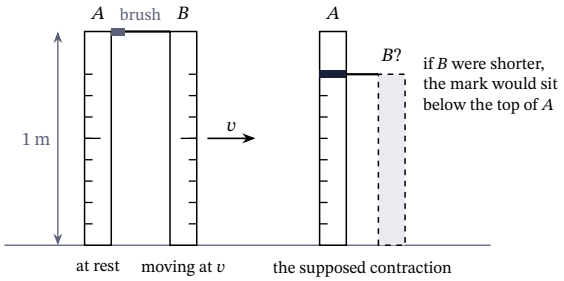


Figure 6. Two identical metre sticks at right angles to the motion, A at rest and B moving past it at speed v with a brush fixed to its top. If motion shortened B, its brush would mark A below the top; seen from B, the same reasoning would make A the shorter stick and the brush would miss. The mark is either there or not, so neither stick can be shorter, and transverse lengths are the same in both frames.

This is why the light clock was built with its mirrors across the motion. The mirror distance d is the same in both frames, and the derivation of time dilation used that fact without comment. Had the pulse bounced along the direction of motion instead, the mirror distance itself would have contracted, and the clock would still tick slowly by the same γ, but the calculation would have had to carry both effects at once. Unit 4 puts both effects, together with the relativity of simultaneity, into one set of four equations.

CHECK YOUR UNDERSTANDING

Question 4

A cylinder of proper length 10 m and proper diameter 2 m flies past at $0.8c$ along its axis. What does a frame at rest measure?

(A) A length of 6 m and a diameter of 1.2 m

(B) A length of 10 m and a diameter of 1.2 m

(C) A length of 6 m and a diameter of 2 m

(D) A length of 6 m and a diameter of 3.3 m

The solution is at the end of the unit.

5.2 Measuring against seeing

Length contraction is a statement about measurement, and measurement in this unit always means recording the positions of the two ends at the same time in the measuring frame, with the frame's own synchronized clocks and rulers. What the eye or a camera would see is a different question. A photograph collects the light that arrives at the lens at one instant, and that light left the near parts of a moving object later than the far parts, because it had less far to travel. For an object moving at a sizeable fraction of c the delays matter, and the picture shows the object's parts where they were at different times. For a sphere the result is still a sphere, and for a cube the result looks like a cube turned through an angle rather than a cube squashed along its motion. Nothing in that contradicts the contraction, which is still there in the measured positions; the camera measures something else. The effect was worked out in 1959, more than fifty years after Einstein's paper, by James Terrell and Roger Penrose independently; a short paper of 1924 by Anton Lampa (1868-1938) had found it for a moving rod and had been forgotten. Unit 5 explains the effect with the geometry of spacetime and shows the first photographs of it, which were taken in 2024 with a trick that stands in for the speed of light and published in 2025.

The length of the moving train is measured without any photograph at all. Line the platform with clocks, synchronized in the platform frame, and put a marker at each clock. At the agreed instant every marker records whether the train is in front of it or not, and the markers that saw the train span the measured length L. No light had to travel anywhere, because each marker looked only at its own spot. A single camera at one point does the opposite, since it combines light that left the far end of the train earlier than the light from the near end, and so its picture mixes positions from different times. The length contraction of this unit is the first kind of measurement, and every length in the worked examples was measured that way.

KEY POINTS

- Two events simultaneous in one inertial frame are not simultaneous in a frame moving along the line that joins them. The train and the two lightning strikes show this from the second postulate alone.

- The Lorentz factor $\gamma = 1/\sqrt{1 - v^2/c^2}$ is 1.25 at $0.6c$, $5/3$ at $0.8c$, and 10 at $0.995c$; it stays within half a per cent of 1 below $0.1c$.
- A moving clock runs slow: $\Delta t = \gamma\,\Delta\tau$. The proper time $\Delta\tau$ is read by the one clock present at both events, and every other frame measures the longer Δt.
- Each of two observers in relative motion finds the other's clock slow. There is no contradiction, because each comparison uses two synchronized clocks on one side, and the other side does not accept that synchronization.
- Muons that live 2.2 microseconds reach the ground from 15 km up: Frisch and Smith counted 408 per hour at sea level where 27 were expected without time dilation, and stored lithium ions confirm the Lorentz factor to a few parts in a billion.
- A moving rod is shorter along its motion, $L = L_0/\gamma$, while lengths across the motion are unchanged. The proper length L_0 is measured in the rod's rest frame.

Solutions to the questions in the text: 1-A, 2-B, 3-C, 4-C. The online version of this unit explains each answer.